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On Markov–Bernstein-Type Inequalities and Their Applications

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1. INTRODUCTION

Let $\mathscr{P}$ be the set of all polynomials and let $\mathscr{P}_n$ be the subset of $\mathscr{P}$ whose elements have degree not exceeding n.

By $Q(x)$ we denote an even convex, twice differentiable function on $(-\infty, \infty)$ for which $Q'(\infty) = \infty$,

$$x(Q''(x)/Q'(x)) \leqslant c_0 \qquad (-\infty < x < \infty) \tag{1.1}$$

and such that Q'' has increasing tendency for $x > 0$, i.e.,

$$0 \leqslant Q''(x_1) \leqslant (1 + c_1)\, Q''(x_2) \qquad (0 < x_1 < x_2). \tag{1.2}$$

Here c_0, c_1 and in what follows c_2,... are positive numbers depending on the choice of Q only. We put

$$w_Q(x) = \exp\{-Q(x)\} \tag{1.3}$$

and we denote by q_n the (unique) positive solution of the equation

$$q_n Q'(q_n) = n. \tag{1.4}$$

Note that along with $\{q_n\}$ the sequence $\{Q'(q_n) = n/q_n\}$ is increasing so that $1 < q_{2n}/q_n < 2$. Moreover by (1.1),

$$2\,\frac{q_n}{q_{2n}} = \frac{Q'(q_{2n})}{Q'(q_n)} = \exp\left\{\int_{q_n}^{q_{2n}} \frac{Q''(x)}{Q'(x)}\,dx\right\} < \exp\left\{c_0 \int_{q_n}^{q_{2n}} \frac{dx}{x}\right\} = \left(\frac{q_{2n}}{q_n}\right)^{c_0}.$$

Thus $(q_{2n}/q_n)^{c_0+1} > 2$, so that

$$1 < c_2 < q_{2n}/q_n < 2. \tag{1.5}$$

For a measurable g let $\| g \| = \operatorname{ess\,sup}_{-\infty<x<\infty} | g(x)|$. The main result of the present paper is

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THEOREM 1.1. *We have for every* $P_n \in \mathscr{P}_n$,

$$\| w_Q P_n' \| < c_3(n/q_n) \| w_Q P_n \| \qquad (n = 0, 1,...). \tag{1.5}$$

Special cases of (1.5) were proved earlier by us in [3, 4, 6, 7]. Theorem 1.1 is proved in Section 3 after we develop the necessary tools in Section 2. Theorem 1.1 has important applications in approximation theory. In our present paper we deal with one of these applications: the weighted polynomial approximations of the derivative of a function by the derivatives of a sequence of polynomials which approximate the function itself, see Section 4. A further important application of Theorem 1.1 to converse theorems of weighted polynomial approximation we intend to treat in a subsequent paper. It will be shown there that (1.5) can be extended to $\mathscr{L}_p$-norms.[1]

As a last application of our result we give in Section 5 lower estimates for the distance of consecutive zeros of certain orthogonal polynomials.

2. LEMMATA

We call w a weight function if $w(x) \geqslant 0$ $(-\infty < x < \infty)$, $x^m w(x) \in \mathscr{L}$ $(m = 0, 1,...)$ and $\int_{-\infty}^{\infty} w\, dx > 0$. For an arbitrary weight function w we denote by $\{p_n(w; x)\}$ the sequence of orthogonormal polynomials with respect to w (see [2]). We observe that $w_Q^\alpha = \exp\{-\alpha Q\}$ is a weight function for every $\alpha > 0$. In fact, we have for $|x| > q_r$

$$\begin{aligned} Q(x) = Q(|x|) &= Q(q_r) + \int_{q_r}^{|x|} Q'(t)\, dt > Q(q_r) + \int_{q_r}^{|x|} (r/t)\, dt \\ &= Q(q_r) + r \log(|x|/q_r), \end{aligned}$$

i.e.,

$$w_Q(x) < w_Q(q_r)(q_r/|x|)^r \qquad (|x| > q_r), \tag{2.1}$$

and this implies that $x^m[w_Q(x)]^\alpha \in \mathscr{L}$ for every nonnegative integer m.

Let now w be an arbitrary weight function and let $\Phi\colon \mathscr{P} \to R$ be a linear functional on $\mathscr{P}$. We introduce the expressions

$$\lambda_n(w; \Phi) = \min_{\tau \in \mathscr{P}_{n-1}} [\Phi(\tau)]^{-2} \int \tau^2 w\, dx, \tag{2.2}$$

where τ runs through all the elements of $\mathscr{P}_{n-1}$ for which $\Phi(\tau) \neq 0$. In (2.2) and in all what follows, whenever lower and upper bounds of an integral are not marked the integration must be extended over the whole real line R.

[1] See note added in proof at the end of paper.

LEMMA 2.1. *We have*

$$\lambda_n^{-1}(w; \Phi) = \sum_{k=0}^{n-1} \{\Phi[p_k(w)]\}^2. \tag{2.3}$$

Remark. This lemma is well known in the special case when Φ is the evaluation functional, i.e., $\Phi(\tau) = \tau(x)$ for some fixed $x \in R$; see [1, Theorem 11.3.1; 2, Theorem 1.4.1].

Proof of Lemma 2.1. We can express an arbitrary $\tau \in \mathscr{P}_{n-1}$ in the form

$$\tau(x) = \sum_{k=0}^{n-1} a_k p_k(w; x).$$

We obtain

$$\left\{\sum_{k=0}^{n-1} a_k \Phi[p_k(w)]\right\}^2 \leqslant \sum_{k=0}^{n-1} a_k^2 \sum_{k=0}^{n-1} \{\Phi[p_k(w)]\}^2 = \int \tau^2 w\, dx \sum_{k=0}^{n-1} \{\Phi[p_k(w)]\}^2 \tag{2.4}$$

and the sign of equality is valid in (2.4) for $a_k = \Phi[p_k(w)]$ $(k = 0, 1,..., n-1)$. Q.E.D.

LEMMA 2.2. *Let w_1 and w_2 be two weight functions for which*

$$w_1(x) \leqslant w_2(x) \qquad (-\infty < x < \infty). \tag{2.5}$$

Then we have for every linear functional Φ

$$\lambda_n(w_1; \Phi) \leqslant \lambda_n(w_2; \Phi). \tag{2.6}$$

This is clearly a consequence of the definition (2.2).

LEMMA 2.3. *Let $w_{Qn}(x) = w_Q(x)$ for $|x| \leqslant q_n$ and $w_{Qn}(x) = 0$ otherwise, thus $w_Q^2(x) \geqslant w_{Qn}^2(x)$; then we have for $n \geqslant c_4$ and $|x| \geqslant q_{2n}$,*

$$\sum_{k=0}^{n-1} [p_k(w_Q^2; x)]^2 \leqslant \sum_{k=0}^{n-1} [p_k(w_{Qn}^2; x)]^2 \leqslant c_4 e^{c_5 n} (q_{2n}/|x|)^{2n} w_Q^{-2}(x). \tag{2.7}$$

Proof. The first half of (2.7) is a consequence of Lemma 2.2. The second half was proved in [9] as Lemma 2.4.

LEMMA 2.4. *There exist numbers c_7 and c_8 so that for every $n > c_7$ and every $\xi \in [-c_8 q_n, c_8 q_n]$ there exists a polynomial of degree n, $r_n(x) = r_n(w_Q, \xi; x)$ for which we have*

$$r_n(x) \leqslant 2 w_Q(x) \qquad (|x| < c_8 q_n), \tag{2.8}$$

$$r_n(\xi) = w_Q(\xi), \tag{2.9}$$

and

$$r_n'(\xi) = w_Q'(\xi). \tag{2.10}$$

Remark. Under more restrictive conditions with respect to Q this Lemma was proved in [9] as Lemma 3.2.

Proof. For $|x| \leqslant q_n$ we have by (1.1) and (1.2) using the fact that Q'' is even, $Q''(x) \leqslant (1 + c_1) Q''(q_n) \leqslant (1 + c_1) c_0(Q'(q_n)/q_n) = (1 + c_1) c_0(n/q_n^2)$; the last step we obtained from (1.4). We infer that for $|x| \leqslant q_n$ and $|\xi| \leqslant q_n$

$$Q(x) \leqslant Q(\xi) + (x - \xi) Q'(\xi) + \tfrac{1}{2}(1 + c_1) c_0(n/q_n^2)(x - \xi)^2 \overset{\text{def}}{=} Q(\xi) + \Psi_\xi(x). \tag{2.11}$$

We observe that $|\xi| \leqslant q_n$ implies $|Q'(\xi)| \leqslant Q'(q_n) = n/q_n$. Thus we can choose $1 > c_8 > 0$ so small that we have

$$|\Psi_\xi(x)| \leqslant 10^{-4}n \qquad (|x| \leqslant c_8 q_n, |\xi| \leqslant c_8 q_n). \tag{2.12}$$

Let $\nu = [n/2]$ and

$$s_\nu(u) = \sum_{r=0}^{\nu} (u^r/r!), \tag{2.13}$$

thus for $n > c_7$ (i.e., for large ν)

$$\tfrac{1}{2} < e^{-u} s_\nu(u) < 2 \qquad (|u| \leqslant \nu/10). \tag{2.14}$$

By (2.11), (2.12), and (2.14)

$$r_n(x) \overset{\text{def}}{=} e^{-Q(\xi)} s_\nu[-\Psi_\xi(x)] \leqslant 2e^{-Q(x)} \qquad (|x| \leqslant c_8 q_n, |\xi| \leqslant c_8 q_n). \tag{2.15}$$

Since $\Psi_\xi(\xi) = 0$ and $\Psi_\xi'(\xi) = Q'(\xi)$ we see that $r_n(x)$ satisfies (2.9) and (2.10). Moreover, (2.8) is implied by (2.15) and (1.3). Finally, as a consequence of (2.13) and $\nu = [n/2]$ we have $r_n \in \mathscr{P}_n$. Q.E.D.

We consider now the sequence of orthonormal polynomials $\{p_n(w_Q^2; x)\}$ with respect to the weight w_Q^2.

LEMMA 2.5. *We have, provided that Q satisfies the conditions stated in the introduction, for every real ξ and every natural n*

$$K_n(w_Q^2; \xi) \overset{\text{def}}{=} \sum_{k=0}^{n-1} [p_k(w_Q^2; x)]^2 \leqslant c_9(n/q_n)\, w_Q^{-2}(\xi). \tag{2.16}$$

Remark. Equation (2.16) was proved in our lecture [9] under the additional condition $Q''(2t) > (1 + c_{11}) Q''(t)$ $(t > c_{12})$ and for the weights $(1 + x^2)^{\beta/2} e^{-x^2/2}$ $(\beta \leqslant 0)$ in our paper [5].

Proof. First let us assume that $|\xi| \leqslant \frac{1}{2}c_8 q_n$. We have by Lemma 2.1 as applied to the functional $\phi(P) = P(\xi)$

$$\begin{aligned}[K_n(w_Q{}^2;\xi)]^{-1} &= \min_{P\in\mathscr{P}_{n-1}} [P(\xi)]^{-2}\int P^2 w_Q{}^2\,dx \\ &\geqslant \min_{P\in\mathscr{P}_{n-1}} [P(\xi)]^{-2}\int_{-c_8q_n}^{c_8q_n} P^2 w_Q{}^2\,dx \\ &\geqslant \tfrac{1}{4}\min_{P\in\mathscr{P}_{n-1}} [P(\xi)]^{-2}\int_{-c_8q_n}^{c_8q_n} (Pr_n)^2\,dx \\ &= \tfrac{1}{4}\min_{P\in\mathscr{P}_{n-1}} [(Pr_n)(\xi)]^{-2}\int_{-c_8q_n}^{c_8q_n} (Pr_n)^2\,dx[w_q(\xi)]^2. \quad (2.17)\end{aligned}$$

In the last two steps we used (2.8) and (2.9). Since $\varphi = Pr_n \in \mathscr{P}_{2n-1}$ we obtain from (2.18) and the transformation $x = c_8 q_n t$

$$\begin{aligned}[K_n(w_Q{}^2;\xi)]^{-1} &\geqslant \min_{Q\in\mathscr{P}_{2n-1}} [\varphi(\xi)]^{-2}\int_{-c_8q_n}^{c_8q_n} \varphi^2\,dx[w_Q(\xi)]^2 \\ &= c_8q_n \min_{\varphi\in\mathscr{P}_{2n-1}} [\varphi(\xi/c_8q_n)]^{-2}\int_{-1}^{1} \varphi^2\,dx[w_2(\xi)]^2 \\ &\geqslant c_{10}(q_n/n)[w_Q(\xi)]^2 \qquad (|\xi| \leqslant \tfrac{1}{2}c_8q_n). \quad (2.18)\end{aligned}$$

For the last step see, e.g., [2, Theorem 3.3, Chap. III]. This proves (2.16) for $|\xi| \leqslant \frac{1}{2}c_8 q_n$ and we know from Lemma 2.3 that it holds also for $|\xi| \geqslant e^{c_6} q_{2n}$, thus it holds by (1.5) for $|\xi| \geqslant c_{11} q_n$. We fill the gap $\frac{1}{2}c_8 q_n < |\xi| < c_{11} q_n$ as follows: In virtue of (1.5) we can find a sufficiently great natural number r so that we will have $q_{rn}/q_n > 2c_{11}/c_8$ so that $|\xi| < c_{11} q_n$ implies $|\xi| < \frac{1}{2}c_8 q_{rn}$ and consequently, by (2.18) as applied to rn in place of n

$$K_n(w_Q{}^2;\xi) \leqslant K_{rn}(w_Q{}^2;\xi) \leqslant c_{10}^{-1}(rn/q_{rn})[w_Q(\xi)]^{-2} \leqslant rc_{10}^{-1}(n/q_n)[w_Q(\xi)]^{-2}.$$

Consequently, (2.16) is valid for every real ξ. Q.E.D.

Let us consider now the polynomials $p_k(w_{Qn}; x)$ and $p_k'(w_{Qn}; x)$ (see Lemma 2.3).

We denote the coefficient of x^k in $p_k(w_{Qn})$. by $\gamma_k(w_{Qn})$. Let the zeros of $p_k(w_{Qn}^2)$ be in decreasing order $x_{\nu k}$ ($\nu = 1, 2,..., k$) and the zeros of $p_k'(w_{Qn}^2)$ in decreasing order we denote by $\xi_{\mu k}$ ($\mu = 1,2,..., k-1$). It is well known that all zeros $x_{\nu k}$ are real and simple and they all are situated in the interval of support $(-q_n, q_n)$. By Rolle's theorem

$$x_{\mu+1,k} < \xi_{\mu k} < x_{\mu k}. \quad (2.19)$$

Since the weight w_{Qn} is even, $p_k(w_{Qn})$ is even for even k and odd for odd k. Let k be odd and $|x| \geqslant 2q_n$, then by (2.19)

$$\begin{aligned} p_k'(w_{Qn}^2;\, x) &= k\gamma_k(w_{Qn}^2) \prod_{\xi_{\mu k}>0} (x^2 - \xi_{\mu k}^2) \\ &\leqslant k\gamma_k(w_{Qn}^2) \prod_{0\leqslant x_{\mu+1,k}<x_{1k}} (x^2 - x_{\mu+1,k}^2) \\ &= k\frac{x}{x^2 - x_{1n}^2} p_k(w_{Qn}^2;\, x) \leqslant k\frac{x}{x^2 - q_n^2} p_k(w_{Qn}^2;\, x) \\ &\leqslant (k/q_n)\, p_k(w_{Qn}^2;\, x) \end{aligned}$$

and by a similar argument the inequality

$$p_k'(w_{Qn}^2;\, x) \leqslant (k/q_n)\, p_k(w_{Qn}^2;\, x) \qquad (x > 2q_n) \tag{2.20}$$

is valid also for even values k; thus it holds for every natural k.

LEMMA 2.6. *We have for sufficiently great* c_{12}

$$\sum_{k=0}^{n-1} [p_k'(w_Q{}^2;\, \xi)]^2 \leqslant \sum_{k=0}^{n-1} [p_k'(w_{Qn}^2;\, \xi)]^2 \leqslant c_{13} w_Q^{-2}(\xi) \qquad (|\xi| > c_{12}q_n). \tag{2.21}$$

Proof. The first half of (2.21) we obtain by applying Lemma 2.2 with the functional $\phi_1(f) = f'(\xi)$. The second half of (2.21) is a consequence of (2.20) and (2.7). Q.E.D.

LEMMA 2.7. *If Q satisfies the conditions stated in the introduction we have for every real ξ and every natural n*

$$K_n'(w_Q{}^2;\, \xi) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} [p_k'(w_Q{}^2;\, \xi)]^2 \leqslant c_{14}(n/q_n)^3 [w_Q(\xi)]^{-2}. \tag{2.22}$$

Proof. By Lemmas 2.1 and 2.4

$$\begin{aligned} [K_n'(w_Q{}^2;\, \xi)]^{-1} &= \min_{P\in\mathscr{P}_{n-1}} [P'(\xi)]^{-2} \int P^2 w_Q{}^2\, dx \\ &\geqslant \min_{P\in\mathscr{P}_{n-1}} [P'(\xi)]^{-2} \int_{-c_8q_n}^{c_8q_n} P^2 w_Q{}^2\, dx \\ &\geqslant \tfrac{1}{4} \min_{P\in\mathscr{P}_{n-1}} [P'(\xi)]^{-2} \int_{-c_8q_n}^{c_8q_n} (Pr_n)^2\, dx. \end{aligned} \tag{2.23}$$

We set $P(x)\, r_n(x) = w_Q(\xi)\, \Psi(x)$. Clearly $\Psi \in \mathscr{P}_{2n-1}$ and by (2.10) and (2.9) we have $P'(\xi) = \Psi'(\xi) + Q'(\xi)\, \Psi(\xi)$.

The last "min" expression in (2.23) is decreased if we allow for the concurrence every polynomial of degree not exceeding $2n - 1$ and not just polynomials divisible by r_n; thus

$$[w_Q(\xi)]^{-2}[K_n'(w_Q^2; \xi)]^{-1} \geqslant \tfrac{1}{4} \min_{\Psi \in \mathscr{P}_{2n-1}} [\Psi'(\xi) + Q'(\xi)\,\Psi(\xi)]^{-2} \cdot \int_{-c_8 q_n}^{c_8 q_n} [\Psi(x)]^2\, dx. \qquad (2.24)$$

Combining it with (2.2) we see that the last minimum expression is of the form $\lambda_n(w_n; \phi_\xi)$ where $w_n(x) = 1$ for $|x| \leqslant c_8 q_n$, $w_n(x) = 0$ otherwise, and $\phi_\xi(f) = f'(\xi) + Q'(\xi) f(\xi)$. By Lemma 2.1 we infer from (2.24)

$$[K_n'(w_Q^2; \xi)]^{-1} \geqslant \tfrac{1}{4}\lambda_n(w_n; \phi_\xi) = \tfrac{1}{4}\left\{\sum_{k=0}^{n-1} [p_k'(w_n; \xi) + Q'(\xi)\, p_k(w_n; \xi)]^2\right\}^{-1}. \qquad (2.25)$$

An elementary calculation shows that

$$p_k(w_n; x) = \frac{1}{(c_8 q_n)^{1/2}} \left(\frac{2k+1}{2}\right)^{1/2} P_k\left(\frac{x}{c_8 q_n}\right)$$

where P_k is the kth degree Legendre polynomial. Standard estimates on P_k and P_k' (e.g., [1, Theorem 7.3.3 resp. Theorem 7.32.4]) show that for $|\xi| \leqslant \frac{1}{2} c_8 q_n$ we have $|p_k(w_n; \xi)| \leqslant c_{14} q_n^{-1/2}$ and $|p_k'(w_n; \xi)| \leqslant c_{15} n q_n^{-3/2}$. Moreover, $|\xi| \leqslant \frac{1}{2} c_8 q_n < q_n$ implies $|Q'(\xi)| \leqslant n/q_n$; thus

$$|p_k'(w_n; \xi) + Q'(\xi)\, p_k(w_n; \xi)| \leqslant c_{16} n q_n^{-3/2} \qquad (|\xi| \leqslant \tfrac{1}{2} c_8 q_n). \qquad (2.26)$$

Equations (2.25) and (2.26) together prove that (2.22) holds under the assumption $|\xi| < \frac{1}{2} c_8 q_n$. As a consequence of Lemma 2.6, (2.22) is also valid if $|\xi| > c_{12} q_n$. The gap corresponding to the values $\frac{1}{2} c_8 q_n \leqslant \xi \leqslant c_{12} q_n$ can be filled in by the same argument as that in the last part of the proof of Lemma 2.5, i.e., replacing n by rn and taking r sufficiently large but fixed. Q.E.D.

In concluding this section we mention that by virtue of Lemma 2.7 of our lecture note [9], the leading coefficients $\gamma_\nu(w_Q)$ of $p_\nu(w_Q)$ satisfy

$$\gamma_{\nu-1}(w_Q)/\gamma_\nu(w_Q) \leqslant c_{17} q_{2\nu} \leqslant 2 c_{17} q_\nu \qquad (\nu = 1, 2, \ldots). \qquad (2.27)$$

3. Proof of the Markov–Bernstein-Type Inequality

Let f be a measurable function for which $w_Q\,|f|$ is essentially bounded, i.e., $\|w_Q f\| < \infty$. Thus we can expand f in the series

$$f(x) \sim \sum_{\nu=0}^{\infty} a_\nu(w_Q^2; f)\, p_\nu(w_Q^2; x) \tag{3.1}$$

with

$$a_\nu(w_Q^2; f) = \int f(t)\, p_\nu(w_Q^2; t)\, w_Q^2(t)\, dt \qquad (\nu = 0, 1, \ldots). \tag{3.2}$$

We denote the sum of the first m terms of (3.1) by $s_m(w_Q^2; f; x)$. We have

$$s_m(w_Q^2; f; x) = \int K_m(w_Q^2; x, t)\, f(t)\, w_Q^2(t)\, dt, \tag{3.3}$$

where in consequence of the Christoffel–Darboux formula (see e.g., [1] or [2])

$$\begin{aligned} K_m(w_Q^2; x, t) &= \sum_{\nu=0}^{m-1} p_\nu(w_Q^2; x)\, p_\nu(w_Q^2; t) \\ &= \frac{\gamma_{m-1}(w_Q^2)}{\gamma_m(w_Q^2)} \cdot \frac{p_m(w_Q^2; x)\, p_{m-1}(w_Q^2; t) - p_{m-1}(w_Q^2; x)\, p_m(w_Q^2, t)}{x - t}. \end{aligned} \tag{3.4}$$

By differentiation we obtain

$$s_m'(w_Q^2; f; x) = \int K_m^{(1,0)}(w_Q^2; x, t)\, f(t)\, w_Q^2(t)\, dt \tag{3.5}$$

with

$$\begin{aligned} &K_m^{(1,0)}(w_Q^2; x, t) \\ &\quad= \sum_{\nu=0}^{m-1} p_\nu'(w_Q^2; x)\, p_\nu(w_Q^2; t) \\ &\quad= \frac{\gamma_{m-1}(w_Q^2)}{\gamma_m(w_Q^2)} \left[\frac{p_m'(w_Q^2; x)\, p_{m-1}(w_Q^2; t) - p_{m-1}'(w_Q^2; x)\, p_m(w_Q^2; t)}{x - t} \right. \\ &\qquad \left. - \frac{p_m(w_Q^2; x)\, p_{m-1}(w_Q^2; t) - p_{m-1}(w_Q^2; x)\, p_m(w_Q^2; t)}{(x - t)^2} \right]. \end{aligned} \tag{3.6}$$

Theorem 3.1. *We have as a consequence of* (2.16), (2.22), *and* (2.27)

$$(1/n) \sum_{m=1}^{n} |\, s_m'(w_Q^2; f; x)|\, w_Q(x) \leqslant c_{18}(n/q_n)\, \|w_Q f\|. \tag{3.7}$$

Proof. Let $I_n = [x - (q_n/n), x + (q_n/n)]$, $J_n = (-\infty, \infty)\text{-}I_n$,

$$f_1(x) = \begin{cases} f(x) & (x \in I_n), \\ 0 & (x \in J_n), \end{cases} \qquad \text{resp.} \qquad f_2(x) = \begin{cases} 0 & (x \in I_n), \\ f(x) & (x \in J_n), \end{cases} \tag{3.8}$$

i.e., $f = f_1 + f_2$, and consequently

$$s_m'(w_Q^2; f; x) = s_m'(w_Q^2; f_1; x) + s_m'(w_Q^2; f_2; x). \tag{3.9}$$

The estimate of the first term is simple: Taking $m \leqslant n$ and Lemma 2.7 in consideration,

$$\begin{aligned} | s_m'(w_Q^2; f_1; x)| &= \int_{x-(q_n/n)}^{x+(q_n/n)} f(t)\, K_m^{(1,0)}(w_Q^2; x, t)\, w_Q^2(t)\, dt \\ &\leqslant \| w_Q f \| \cdot \left\{ 2(q_n/n) \int [K_m^{(1,0)}(w_Q^2; x, t)]^2 w_Q^2(t)\, dt \right\}^{1/2} \\ &= \left\{ 2(q_n/n) \sum_{\nu=0}^{m-1} [p_\nu'(w_Q^2; x)]^2 \right\}^{1/2} \| w_Q f \| \\ &\leqslant c_{19}(n/q_n) \| w_Q f \| [w_Q(x)]^{-1}. \end{aligned} \tag{3.10}$$

In order to estimate the contribution of the $s_m(w_Q^2; f_2; x)$ we introduce the auxiliary functions

$$\mathscr{F}_n(t) = \frac{f_2(t)}{x - t}, \qquad \mathscr{G}_n(t) = \frac{f_2(t)}{(x - t)^2}. \tag{3.11}$$

By Bessel's inequality the coefficients of the orthogonal expansion (3.1) of $\mathscr{F}_n$ resp. $\mathscr{G}_n$ satisfy, in consequence of (3.8) and (3.11),

$$\sum_{m=0}^{\infty} [a_m(w_Q^2; \mathscr{F}_n)]^2 \leqslant \int \mathscr{F}_n^2 w_Q^2\, dt \leqslant [\| w_Q f \|]^2 \int_{J_n} \frac{dt}{(x - t)^2} = 2 \frac{n}{q_n} [\| w_Q f \|]^2, \tag{3.12}$$

resp.

$$\sum_{m=0}^{\infty} [a_m(w_Q^2; \mathscr{G}_n)]^2 \leqslant [\| w_Q f \|]^2 \int_{J_n} \frac{dt}{(x - t)^4} = \frac{2}{3} \left(\frac{n}{q_n}\right)^3 [\| w_Q f \|]^2. \tag{3.13}$$

Following an idea of T. Carleman (see [2]), we have by (3.5) and (3.6)

$$\begin{aligned} s_m'(w_Q^2; f_2; x) = \frac{\gamma_{m-1}(w_Q^2)}{\gamma_m(w_Q^2)} \{ & p_m'(w_Q^2; x)\, a_{m-1}(w_Q^2; \mathscr{F}_n) \\ & - p_{m-1}'(w_Q^2; x)\, a_m(w_Q^2; \mathscr{F}_n) - p_m(w_Q^2; x)\, a_{m-1}(w_Q^2; \mathscr{G}_n) \\ & + p_{m-1}(w_Q^2; x)\, a_m(w_Q^2; \mathscr{G}_n)\}. \end{aligned} \tag{3.14}$$

Consequently by (2.27),

$$\frac{1}{n}\sum_{1}^{n} | s_m'(w_Q^2; f_2 ; x)|$$

$$\leqslant 2c_{17} \frac{q_n}{n} \sum_{m=1}^{n} \{| p_m'(w_Q^2; x)| \, | a_{m-1}(w_Q^2; \mathscr{F}_n)|$$
$$+ | p'_{m-1}(w_Q^2; x)| \, | a_m(w_Q^2; \mathscr{F}_n)| + | p_m(w_Q^2; x)| \, | a_{m-1}(w_Q^2; \mathscr{G}_n)|$$
$$+ | p_{m-1}(w_Q^2; x)| \, | a_m(w_Q^2; \mathscr{G}_n)|\}$$
$$\leqslant 4c_{17} \frac{q_n}{n} \left\{\left(\sum_{0}^{n} [p_m'(w_Q^2; x)]^2\right)^{1/2}\left(\sum_{0}^{\infty} [a_m(w_Q^2; \mathscr{F}_n)]^2\right)^{1/2}\right.$$
$$\left. + \left(\sum_{0}^{n} [p_m(w_Q^2; x)]^2\right)^{1/2}\left(\sum_{0}^{\infty} [a_m(w_Q^2; \mathscr{G}_n)]^2\right)^{1/2}\right\}.$$

Inserting in the last expression the estimates (2.22) and (3.12), resp. (2.16) and (3.13), we obtain

$$\frac{1}{n}\sum_{1}^{n} | s_m'(w_Q^2; f_2 ; x)| \leqslant c_{20} \frac{n}{q_n} \| w_Q f \| \, [w_Q(x)]^{-1}. \tag{3.15}$$

We see from (3.9), (3.10), and (3.15) that (3.7) holds true. Q.E.D.

THEOREM 3.2. *Under the conditions of Theorem* 3.1 *we have for every* $P_n \in \mathscr{P}_n$

$$\| w_Q P_n' \| \leqslant 4C_{18}(n/q_n) \, \| w_Q P_n \|. \tag{3.16}$$

Remark. We have proved in Lemma 2.5, Lemma 2.7 resp. concerning (2.27) in [9] that our assumptions (2.16), (2.22), and (2.27) are satisfied provided that Q is convex twice differentiable $Q'(\infty) = \infty$ and it satisfies (1.1) and (1.2). It follows that *Theorem* 1.1 *is implied by Theorem* 3.2. In turn our assumptions do hold also for weights which are more general.

Proof of Theorem 3.2. In consequence of the evident relation $s_m(w_Q^2; P_n ; x) = P_n(x)$ $(m = n + 1, n + 2,...)$, valid for every $m > n$ and every $P_n \in \mathscr{P}_n$, the shifted de la Vallée Poussin means

$$v_n(w_Q^2; f; x) = (1/n) \sum_{n+1}^{2n} s_m(w_Q^2; P_n ; x) \tag{3.17}$$

satisfy

$$v_n(w_Q^2; P_n ; x) = P_n(x) \qquad (P_n \in \mathscr{P}_n) \tag{3.18a}$$

and

$$v_n'(w_Q^2; P_n ; x) = P_n'(x) \qquad (P_n \in \mathscr{P}_n). \tag{3.18b}$$

Thus by (3.18b), (3.17), and Theorem 3.1 for every $P_n \in \mathscr{P}_n$

$$\begin{aligned} w_Q(x) \mid P_n'(x)| &= \left| (1/n) \sum_{n+1}^{2n} s_m'(w_Q^2; P_n; x) \right| w_Q(x) \\ &\leqslant 2(1/2n) \sum_{1}^{2n} | s_m'(w_Q^2; P_n; x)| w_Q(x) \\ &\leqslant 2c_{18}(2n/q_{2n}) \| w_Q P_n \| \leqslant 4c_{18}(n/q_n) \| w_Q P_n \|. \end{aligned}$$

Thus $\| w_Q P_n' \| \leqslant 4c_{18}(n/q_n) \| w_Q P_n \|$. Q.E.D.

In concluding this section let us observe that (3.16) and (2.16) together imply (2.22), which in turn was used to prove (3.16). Indeed it follows from (2.16) by Schwartz's inequality that the expression

$$K_n(w_Q^2; x; y) = \sum_{0}^{n-1} p_\nu(w_Q^2; x)\, p_\nu(w_Q^2; y),$$

which is a polynomial of degree n in x for y fixed and vice versa, satisfies

$$| K_n(w_Q^2; x, y)| \, w_Q(x)\, w_Q(y) \leqslant C_9(n/q_n). \tag{3.19}$$

We apply to (3.19) r-times the inequality (3.16) with respect to the variable x and r-times with respect to the variable y and infer that the expressions

$$K_n^{(r,r)}(w_Q^2; x, y) = \sum_{0}^{n-1} p_\nu^{(r)}(w_Q^2; x)\, p_\nu^{(r)}(w_Q^2; y) \tag{3.20}$$

satisfy the inequalities

$$| K_n^{(r,r)}(w_Q^2; x, y)| \, w_Q(x)\, w_Q(y) \leqslant C_9(4C_{18})^{2r}(n/q_n)^{2r+1}. \tag{3.21}$$

By setting $x = y$ we get

$$\sum_{0}^{n-1} [p_\nu^{(r)}(w_Q^2; x)]^2 \leqslant c_9 e^{c_{21} r}(n/q_n)^{2r+1}[w_Q(x)]^{-2}. \tag{3.22}$$

Clearly, (2.16) is the special case $r = 1$ of (3.22). Note that we proved the validity of (3.22) under the conditions (2.16), (2.22), and (2.27) which are satisfied under our assumptions concerning Q formulated in the Introduction.

4. On Simultaneous Approximation

We start this section by compiling some earlier results which we are going to apply.

LEMMA 4.1. *The partial sums* $s_m(w_Q^2; f; x)$ *of the orthogonal expansion* (3.1) *satisfy*

$$(1/n) \sum_{m=1}^{n} |s_m(w_Q^2; f; x)| \, w_Q(x) \leqslant c_{21} \| w_Q f \| \qquad (-\infty < x < \infty). \tag{4.1}$$

Proof. This is a consequence of (2.16) and (2.27) and is proved along the lines of the proof of Theorem 3.1; see [9, Theorem 4.1].

By virtue of Lemma 4.1 the shifted de la Vallée Poussin means (3.17) has the property that

$$\| v_n(w_Q^2; f) \, w_Q \| \leqslant 2(1/2n) \sum_{1}^{2n} | s_m(w_Q^2; f; x)| \, w_Q(x) \leqslant 2c_{21} \| w_Q f \|. \tag{4.2}$$

Let

$$\epsilon_n(w_Q; f) \stackrel{\text{def}}{=} \inf_{P \in \mathscr{P}_n} \|(f - P) \, w_Q \|. \tag{4.3}$$

and $P_n \in \mathscr{P}$ be such that

$$\|(f - P_n) w_Q \| \leqslant 2\epsilon_n(w_Q; f). \tag{4.4}$$

By (3.18a) we obtain from (4.2) and (4.3)

$$\begin{aligned} \|[f - v_n(w_Q^2; f)] w_Q\| &\leqslant \|(f - P_n) w_Q \| + \| v_n(w_Q^2; f - P_n) w_Q \| \\ &\leqslant (1 + 2c_{21}) \|(f - P_n) w_Q \| \leqslant 2(1 + 2c_{21}) \, \epsilon_n(w_Q; f). \end{aligned} \tag{4.5}$$

In the rest of this section we assume that

$$Q'(2x)/Q'(x) > 1 + c_{22} \qquad (x > c_{23}). \tag{4.6}$$

Let us observe that (4.6) does hold under the condition that

$$x(Q''(x))/(Q'(x)) > c_{24} \qquad (x > c_{23}). \tag{4.7}$$

LEMMA 4.2. *If* Q *satisfies* (4.6) *besides all the conditions stated in the Introduction then*

$$\lambda_n(w_Q^2; \xi) \stackrel{\text{def}}{=} [K_n(w_Q^2; \xi)]^{-1} \leqslant c_{25}(q_n/n) \, w_Q^2(\xi) \qquad (|\xi| \leqslant c_{26} q_n). \tag{4.8}$$

This was proved in [10].

For an f satisfying $w_Q f \in \mathscr{L}$ we set

$$\epsilon_n^{(1)}(w_Q; f) \stackrel{\text{def}}{=} \inf_{P \in \mathscr{P}_n} \int |f - P| \, w_Q \, dt. \tag{4.9}$$

LEMMA 4.3. *Let* f *be of bounded variation in every finite interval then we have under the same conditions as in Lemma* 4.2

$$\epsilon_n^{(1)}(w_Q; f) \leqslant c_{26}(q_n/n) \int w_Q(t) \, | df(t)|. \tag{4.10}$$

This is obtained by combining results of our paper [8, Theorems 2.2 and 3.1] with Lemma 4.2.

By Δ_n we denote the set of functions g which satisfy $\| w_Q g \| < \infty$ and which are orthogonal to $\mathscr{P}_n$ with respect to the weight w_Q^2, i.e.,

$$\int g P_n w_Q^2 \, dt = 0 \qquad (P_n \in \mathscr{P}_n). \tag{4.11}$$

LEMMA 4.4. *We have*

$$\left\| w_Q(x) \int_0^x g(t) \, dt \right\| \leqslant C_{27}(q_n/n) \, \| g w_Q \| \qquad (g \in \Delta_n). \tag{4.12}$$

Proof. (See [9, Lemma 5.4].) Letting

$$\begin{aligned} \phi_x(t) &= w_Q^{-2}(t) \qquad (t \in [0, x]), \\ &= 0 \qquad\qquad (t \notin [0, x]), \end{aligned}$$

we have for arbitrary $P_n \in \mathscr{P}_n$

$$\begin{aligned} \left| \int_0^x g(t) \, dt \right| &= \left| \int_{-\infty}^{\infty} g(t) \, \phi_x(t) \, w_Q^2(t) \, dt \right| \\ &= \left| \int_{-\infty}^{\infty} g(t)[\phi_x(t) - P_n(t)] \, w_Q^2(t) \, dt \right| \\ &\leqslant \| g w_Q \| \, \epsilon_n^{(1)}(w_Q \, ; \phi_x), \end{aligned} \tag{4.13}$$

and by virtue of Lemma 4.3

$$\epsilon_n^{(1)}(w_Q \, ; \phi_x) \leqslant C_{28}(q_n/n) \, w_Q^{-1}(x). \tag{4.14}$$

Q.E.D.

LEMMA 4.5. *Let $\mathscr{F}$ be absolutely continuous and $\| \mathscr{F}' w_Q \| < \infty$. Then the polynomial*

$$V_n(w_Q^2; \mathscr{F}; x) = \mathscr{F}(0) + \int_0^x v_n(w_Q^2; \mathscr{F}'; t) \, dt \in \mathscr{P}_n \tag{4.15}$$

satisfies

$$\| w_Q[\mathscr{F} - V_n(w_Q^2; \mathscr{F})] \| \leqslant C_{29}(q_n/n) \, \epsilon_n(w_Q \, ; \mathscr{F}'). \tag{4.16}$$

Proof. In consequence of (3.17), $\mathscr{F}' - v_n(w_Q^2; \mathscr{F}') \in \Delta_n$. Consequently, by virtue of Lemma 4.4,

$$\begin{aligned} \| w_Q[\mathscr{F} - V_n(w_Q^2; \mathscr{F})] \| &= \left\| w_Q(x) \int_0^x [\mathscr{F}'(t) - v_n(w_Q^2; \mathscr{F}'; t)] \, dt \right\| \\ &\leqslant c_{27}(q_n/n) \, \| w_Q[\mathscr{F}' - v_n(w_Q^2; \mathscr{F}')] \| \\ &\leqslant c_{29}(q_n/n) \, \epsilon_n(w_Q \, ; \mathscr{F}'). \end{aligned}$$

Q.E.D.

LEMMA 4.6. *We have*

$$\epsilon_n(w_Q\,;\mathscr{F}) \leqslant c_{30}(q_n/n)\,\epsilon_{n-1}(w_Q\,;\mathscr{F}'). \tag{4.17}$$

Proof. We proved this lemma under additional restrictions on Q in [9] (as Lemma 6.1). The proof is the same: Lemma 4.5 implies

$$\epsilon_n(w_Q\,;\mathscr{F}) \leqslant c_{31}(q_n/n)\,\|\,w\mathscr{F}'\,\| \tag{4.18}$$

and we replace in (4.18) $\mathscr{F}'$ by $\mathscr{F}' - P_{n-1}$, where $P_{n-1} \in \mathscr{P}_{n-1}$ satisfies $\|\,w_Q(\mathscr{F}' - P_{n-1})\| \leqslant 2\epsilon_{n-1}(w_Q\,;\mathscr{F}')$.

THEOREM 4.1. *We assume that Q satisfies the conditions of the Introduction and also satisfies* (4.2); *let $\mathscr{F}$ be continuously differentiable and let $P_n \in \mathscr{P}_n$ such that*

$$\|\,w_Q(\mathscr{F} - P_n)\| < \eta_n\,. \tag{4.19}$$

Then

$$\|\,w_Q(\mathscr{F}' - P_n') \leqslant c_{28}(n/q_n)\eta_n + c_{29}\epsilon_{n-1}(w_Q\,;\mathscr{F}'). \tag{4.20}$$

Remark. If $\eta_n < a\epsilon_n(w_Q\,;\mathscr{F})$ we have by (4.20) and (4.17)

$$\begin{aligned}\|\,w_Q(\mathscr{F}' - P_n')\| &\leqslant ac_{28}(n/q_n)\,\epsilon_n(w_Q\,;\mathscr{F}) + c_{29}\epsilon_{n+1}(w_Q\,;\mathscr{F}')\\ &\leqslant (ac_{28}c_{26} + c_{28})\,\epsilon_{n-1}(w_Q\,;\mathscr{F}');\end{aligned} \tag{4.21}$$

i.e., the derived sequence of a sequence which is good approximating to $\mathscr{F}$ with the weight w_Q is again a good approximating sequence to $\mathscr{F}'$ with the same weight.

Proof of Theorem 4.1. In consequence of (4.16) and (4.19)

$$\|\,w_Q[V_n(w_Q{}^2;\mathscr{F}) - P_n]\| \leqslant \eta_n + c_{29}(q_n/n)\,\epsilon_n(w_Q\,;\mathscr{F}'); \tag{4.22}$$

thus by virtue of Theorem 3.2

$$\begin{aligned}\|\,w_Q[V_n'(w_Q{}^2;\mathscr{F}) - P_n']\| &= \|\,w_Q[v_n(w_Q{}^2;\mathscr{F}') - P_n']\|\\ &\leqslant 4c_{18}(n/q_n)[\eta_n + c_{29}(q_n/n)\,\epsilon_n(w_Q\,;\mathscr{F}')].\end{aligned} \tag{4.23}$$

Finally, by (4.5)

$$\begin{aligned}\|\,w_Q(\mathscr{F}' - P_n')\| &\leqslant \|\,w_Q[\mathscr{F}' - v_n(w_Q\,;\mathscr{F}')\| + \|\,w_Q[v_n(w_Q{}^2;\mathscr{F}') - P_n']\|\\ &\leqslant rc_{18}(n/q_n)\eta_n + [4c_{28}c_{29} + 2(1 + 2c_{21})]\,\epsilon_n(w_Q\,;\mathscr{F}').\end{aligned} \tag{4.24}$$

Q.E.D.

5. On the Zeros of Orthogonal Polynomials

Theorem 5.1. *By a proper choice of the positive numbers c_{30}, c_{31}, c_{32} every pair of consecutive zeros x_{rn} and $x_{r+1,n}$ of $p_n(w_Q{}^2; x)$ which are situated in $[-c_{30}q_n, c_{30}q_n]$ satisfies*

$$c_{31}(q_n/n) < x_{rn} - x_{r+1,n} < c_{32}(q_n/n). \tag{5.1}$$

Proof. The second part of the inequality (5.1) was proved in [10]. We can assume without loss of generality that $C_{30} < 1$. The proof of the first part runs as follows: By the Christoffel–Darboux formula we have, taking in consideration that $p_n(w_Q{}^2; x_{rn}) = 0$:

$$K_n(w_Q{}^2; x_{rn}; x) \stackrel{\text{def}}{=} \sum_{\nu=0}^{n-1} p_\nu(w_Q{}^2; x_{rn})\, p_\nu(w_Q{}^2; x)$$

$$= \frac{\gamma_{n-1}(w_Q{}^2)}{\gamma_n(w_Q{}^2)} \frac{p_{n-1}(w_Q{}^2; x_{rn})\, p_n(w_Q{}^2; x)}{x - x_{rn}}. \tag{5.2}$$

We infer from (5.2) that

$$K_n(w_Q{}^2; x_{rn}, x_{r+1,n}) = 0. \tag{5.3}$$

By [10, Theorem 3.1] we have

$$K_n(w_Q{}^2; x_{rn}, x_{rn}) \geqslant c_{33}(n/q_n)\, w_Q^{-2}(x_{rn}). \tag{5.4}$$

In turn, by Lemmas 2.5 and 2.7 we have for every $x \in (-\infty, \infty)$

$$|(d/dx)\{K_n(w_Q{}^2; x_{rn}, x)\}| = \left| \sum_{\nu=0}^{n-1} p_\nu(w_Q{}^2; x_{rn})\, p_\nu'(w_Q{}^2; x) \right|$$

$$\leqslant \left\{ \sum_{\nu=0}^{n-1} [p_\nu(w_Q{}^2; x_{rn})]^2 \cdot \sum_{\nu=0}^{n-1} [p_\nu'(w_Q{}^2; x_{rn})]^2 \right\}^{1/2}$$

$$\leqslant C_{34}(n/q_n)^2[w_Q(x_{rn})\, w_Q(x)]^{-1}. \tag{5.5}$$

Now let $x \in [x_{r+1}, x_{rn}]$; thus by the already established right-hand side of inequality (5.1) $0 < x_{rn} - x < C_{32}(q_n/n)$. Hence

$$|\, Q(x_{rn}) - Q(x)| \leqslant c_{32}(q_n/n) \cdot Q'(c_{30}q_n) \leqslant c_{32}(q_n/n)\, Q'(q_n) = c_{32}$$

so that

$$[w_Q(x)]^{-1} \leqslant c_{35}[w_Q(x_{rn})]^{-1} \qquad (x \in [x_{r+1,n}, x_{rn}]). \tag{5.6}$$

From (5.5) and (5.6)

$$|(d/dx)\{K_n(w_Q{}^2; x_{rn}; x)\}| \leqslant c_{34}(n/q_n)^2[w_Q(x_{rn})]^{-2} \qquad (x \in [x_{r+1,n}, x_{rn}]). \tag{5.7}$$

By (5.3), (5.4), and (5.7) we have

$$c_{33}(n/q_n)[w_Q(x_{rn})]^{-2} \leqslant K_n(w_Q{}^2; x_{rn}, x_{rn}) - K_n(w_Q{}^2; x_{rn}, x_{r+1,n})$$

$$= \int_{x_{r+1,n}}^{x_{rn}} (d/dx)\{K_n(w_Q{}^2; x_{rn}, x)\}\, dx$$

$$\leqslant c_{34}(n/q_n)^2[w_Q(x_{rn})]^{-2}(x_{rn} - x_{r+1,n}),$$

i.e., $x_{rn} - x_{r+1,n} \geqslant c_{33}(c_{34})^{-1}(q\ /n)$. Q.E.D.

Note Added in Proof. We extended inequality (1.5) to $\mathscr{L}_p$-norms and applied it to the weighted polynomial approximation in "Approximation Theory II" (G. G. Lorentz, C. K. Chui, and L. L. Shumaker, Eds.), pp. 369–377, Academic Press, 1976.

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